Faculty Working Papers

ON THE MEASUREMENT ERRORS AND RANKING OF COMPOSITE PERFORMANCE MEASURES

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Summary:

The measurement errors on both market rates of return and zero beta portfolio rates of return can bias the estimated Jensen performance measure, the estimated systematic risk and Treynor performance measure. It is shown that the above mentioned biases caused by the measurement errors of market rates of return and zero beta portfolio rates of return can lead to inconsistent ranking of the portfolio evaluation.

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I. Introduction

The impact of errors in measurement of the market rates of return and the risk-free rate on the estimated beta coefficient of the capital asset pricing model (CAPM) has been studied in detail by Roll (1969) and Lee and Jen (1978). They have shown that the measurement errors generally bias the estimated Jensen's performance measure and the estimated systematic risk. They have also used the biases of both Jensen measure and systematic risk induced by the measurement errors to interpret Friend and Blume's (1970) results, i.e., there exist some evidence on non-zero covariance relationships between the estimated Jensen performance measures and its risky proxy. Specifically, the inconsistency of the bias (non-zero covariance) associated with the estimated composite performance measures in two subperiods can be explained by examining the effect of the measurement errors on the estimated systematic risk, average market rates of return and risk-free rate. However, the effect of measurement errors on the estimates of the parameters associated with the two-factor model has not been explicitly investigated in their study. It is the main purpose of this paper to extend Lee and Jen's study to the two factor model. It is shown that both estimated alpha and beta coefficients are not free from the bias caused by the measurement errors associated with both market rates of return and the return on the minimum-variance zero beta portfolio. It is also shown that the measurement errors of market rates of return and zero-beta factor can lead to a biased representation (or ranking) of the estimated Jensen's and Treynor's performance measures. Furthermore, the effect of the measurement errors induces additional ambiguity shown by Roll (1978) that inconsistent ranking may arise in performance evaluation of portfolios (or assets) when different indices are used as proxies for the market portfolio.

In the second section of this paper, the effect of measurement errors on the estimates of alpha and systematic risk derived from the two-factor model is studied. In the third section, the measurement errors and the estimated Treynor's measure are discussed in detail. In the fourth section, the impact of the measurement errors on Jensen's measure is investigated. Finally, the results of this study are summarized in the fifth section.

II. Measurement Errors and Estimated Alpha and Beta Coefficients

Black (1972) has developed a two-factor model describing the equilibrium structure of security (or portfolio) returns. The model is represented by the following equation:

$$E(R_{jt}) = E(R_{zt}) + \beta_j[E(R_{mt}) - E(R_{zt})]; \qquad (1)$$

where R_{jt} = the return on security (or portfolio) j in period t,

R_{zt} = the return on the minimum-variance zero beta portfolio in period t,

R_{mt} = the return on the market portfolio in period t,

and E indicates the mathematical expectation of a random variable.

In their empirical study, Black, Jensen and Scholes (1972) have shown that the following time-series equation of (1) provides a better description for the behavior of security (or portfolio) returns:

$$R_{jt} - R_{zt} = \alpha_j + \beta_j(R_{mt} - R_{zt}) + \varepsilon_{jt}, \qquad (2)$$

where α_j and β_j are assumed to be stationary over time, and ε_{jt} is assumed to be normally, independently distributed with mean zero and variance σ_ε^2, $NI(0,\sigma_\varepsilon^2)$, and is independent of R_{mt}. The effect of measurement error on the estimates of α_j and β_j associated with the two-factor model in (2) is studied in the following section.

1. Measurement Error on R_{mt}

We first investigate the impact of measurement error on R_{mt} on the systematic risk estimate, $\hat{\beta}_j$. Following Lee and Jen's (1978) results, a model specifying the measurement error on R_{mt} can be defined as [1]

$$R_{mt} = R'_{mt} + \tau + \eta_t, \tag{3}$$

where R'_{mt} = the return on a proxy for the market portfolio, say the return on the New York Stock Exchange price index or Standard and Poor's composite index;

η_t is random measurement error of R_{mt}, and is assumed to be $NI(0,\sigma_\eta^2)$; and τ is a constant.

Using equation (3), the two-factor capital asset pricing model (CAPM) in (2) can be rewritten into the equation

$$R_{jt} - R_{zt} = \alpha_j + \beta_j[(R'_{mt} + \tau + \eta_t) - R_{zt}] + \varepsilon_{jt} \tag{4}$$

where η_t is assumed to be independent of ε_{jt}, R_{jt}, R_{mt}, and R_{zt}. Then the ordinary least-squares (OLS) estimators of β_j and α_j can be written respectively as [2]

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[1] Lee and Jen (1978) have a brief discussion of measurement error on R_{mt} induced by using a market index as a proxy for the market portfolio.

[2] See Appendix (A) for the derivative of equations (5) and (6).

$$\hat{\beta}_j = \beta_j - \frac{\beta_j \sigma_\eta^2}{\sigma_m^2 + \sigma_\eta^2 + \sigma_z^2} + s_j \tag{5}$$

and

$$\hat{\alpha}_j = \alpha_j - (s_j - \frac{\beta_j \sigma_\eta^2}{\sigma_m^2 + \sigma_\eta^2 + \sigma_z^2})(\bar{R}_m - \bar{R}_z), \tag{6}$$

where $\bar{R}_m = \bar{R}'_m + \tau + \bar{\eta}$; the bars indicate the sample averages; σ_m^2 and σ_z^2 are the variances of R_{mt} and R_{zt}, respectively; and s_j, the sample variation of β_j, is assumed to be distributed with mean zero and variance σ_s^2. Equation (5) indicates that the estimated systematic risk, $\hat{\beta}_j$, consists of three components: the ex-ante systematic risk, β_j, the sample variation of β_j, and the bias, $-\frac{\beta_j \sigma_\eta^2}{\sigma_m^2 + \sigma_\eta^2 + \sigma_z^2}$, induced by the measurement error on R_{mt}. If there exists no measurement error on R_{mt} the estimated systematic risk contains β_j and its sampling variation, s_j. This implies that the estimated systematic risk is influenced only by the sampling variation of β_j. However, as equation (5) indicates, the measurement error on R_{mt} will have a significant impact on the estimated systematic risk. The reason is as follows: The estimated systematic risk of a security (or portfolio) with a positive beta coefficient β_j will be biased downward since the bias becomes negative. On the other hand, the estimated systematic risk of a negative-beta-coefficient security (or portfolio) is upward biased, for the bias is positive. Similarly the estimated alpha (or Jensen's performance measure), $\hat{\alpha}_j$, is upward (downward) biased for a security (or portfolio) with a positive (negative) beta coefficient if $(\bar{R}_m - \bar{R}_z)$ is positive.[3] Therefore, the measurement

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[3] A reverse conclusion is expected if $(\bar{R}_m - \bar{R}_z)$ is negative.

error on the market rate of return could lead to an upward or downward bias in the estimated systematic risk and the estimated alpha.

2. Measurement Error on R_{zt}

Black, Jensen and Scholes (1972) (hereinafter, BJS) have derived a technique to measure the return on the minimum-variance zero beta portfolio. The BJS estimate of R_{zt} is defined as [4]

$$R_{zt} = \sum_{j=1}^{N} w_j R_{zjt}, \tag{7}$$

where N = the number of securities (or portfolios) used to estimate R_{zt},

$w_j = (1 - \beta_j)^2 / \sum_{j=1}^{N} (1 - \beta_j)^2$,

R_{zjt} = the estimate of R_{zt} based on the return on security (or portfolio) j

$$= \frac{R_{jt} - \beta_j R_{mt}}{1 - \beta_j} + \varepsilon'_{jt}, \text{ and} \tag{8}$$

$$\varepsilon'_{jt} = \frac{\varepsilon_{jt}}{1 - \beta_j} .$$

Note that R_{zjt} defined in (8) depends on R_{mt}. Whenever R_{mt} is measured with error, the BJS estimator in (7) will not be free from the bias induced by the measurement error on R_{mt}. This can be shown as follows: using equations (3), (7) and (8), a model specifying the measurement error on R_{zt} can be derived as

$$R_{zt} = R'_{zt} + A + \delta_t, \tag{9}$$

[4]BJS have shown that the estimate of R_{zt} defined in (7) is unbiased and approximately efficient with a minimum error variance.

where $R'_{zt} = \sum_{j=1}^{N} w_j\left(\frac{R_{jt} - \beta_j R'_{mt}}{1 - \beta_j} + \varepsilon'_{jt}\right),$

= the weighted average of the estimates of R_{zt} based on the returns on N securities (or portfolios),

$A = \sum_{j=1}^{N} w_j\left(\frac{\beta_j \tau}{1 - \beta_j}\right)$, τ is defined in (3), and

$\delta_t = \sum_{j=1}^{N} w_j\left(\frac{\beta_j \eta_t}{1 - \beta_j}\right)$, is distributed with mean zero and variance σ_δ^2.

If R_{mt} is measured without error, $\tau = 0$ and $\eta_t = 0$. Then, $R_{zt} = R'_{zt}$ for $A = 0$ and $\delta_t = 0$. Substituting equation (9) into equation (4) gives a general form of the two-factor CAPM:

$$R_{jt} - (R'_{zt} + A + \delta_t) = \alpha_j + \beta_j[(R'_{mt} + \tau + \eta_t) - (R'_{zt} + A + \delta_t)] + \varepsilon_{jt}, \quad (10)$$

where $(\eta_t - \delta_t) \sim N(0,\sigma_1^2)$, $\sigma_1^2 = \sigma_\eta^2(1 + c - d)$,[5]

$$c = \sum_{j=1}^{N} \frac{w_j^2\beta_j^2}{(1-\beta_j)^2}, \quad d = \sum_{j=1}^{N} \left(\frac{w_j\beta_j}{1-\beta_j}\right),$$

$E(\eta_t\varepsilon_{jt}) = 0$, $E(\delta_t\varepsilon_{jt}) = 0$, and

η_t and δ_t are independent of R_{mt} and R_{zt}.

Following a similar analysis of deriving equations (5) and (6), the OLS estimators of β_j and α_j in equation (10) can be expressed respectively as[6]

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[5] $Var(\eta_t - \delta_t) = \sigma_\eta^2 + \sigma_\delta^2 - Cov(\eta_t,\delta_t) = \sigma_\eta^2 + \sigma_\eta^2 c - \sigma_\eta^2 d$
$= \sigma_\eta^2(1 + c - d)$ where $\sigma_\delta^2 = Var(\delta_t) = \sigma_\eta^2 c$ and $Cov(\eta_t,\delta_t) = \sigma_\eta^2 d$.

[6] See Appendix (B) for the detailed derivation.

$$\hat{\beta}'_j = \beta_j + \psi_j + s_j \tag{11}$$

and

$$\hat{\alpha}'_j = \alpha_j + (\psi_j - s_j)(\bar{R}_m - \bar{R}_z), \tag{12}$$

where

$$\psi_j = \frac{\sigma_\eta^2[d(1 + 2\beta_j) - \beta_j - c(1 + \beta_j)]}{\sigma_m^2 + \sigma_z^2 + \sigma_1^2} \tag{13}$$

= the effect of measurement errors

$$\bar{R}_z = \bar{R}'_z + A + \bar{\delta}, \quad \bar{R}'_z = \sum_{t=1}^{N} R'_{zt}/n, \text{ and } \bar{\delta} = \sum_{t=1}^{n} \delta_t/n.$$

As indicated by (11), (12), and (13), the measurement errors on both R_{mt} and R_{zt} induces the bias (ψ_j) in estimating β_j and α_j.[7] The behavior of the bias caused by the measurement errors is difficult to analyze. However, it can be concluded that the estimates of β_j and α_j are upward or downward biased, depending on whether the bias, ψ_j, is positive or negative. The β_j and α_j are over-estimated if the bias is positive; otherwise, they are underestimated. The above analysis has indicated that the measurement errors on either R_{mt} or R_{zt} (or both) have a significant impact on the estimated systematic risk and the estimated alpha (or Jensen's measure).[8] As a consequence, the estimated Treynor's and the estimated Jensen's performance measures are affected by the measurement

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[7] If R_{mt} and R_{zt} are not measured with errors ($R_{mt} = R'_{mt}$ and $R_{zt} = R'_{zt}$), equations (11) and (12) become $\hat{\beta}'_j = \beta_j + s_j$ and $\hat{\alpha}'_j = \alpha_j - s_j(\bar{R}_m - \bar{R}_z)$, respectively, where $\bar{R}_m = \sum_{t=1}^{n} R_{mt}/n$ and $\bar{R}_z = \sum_{t=1}^{n} R_{zt}/n$.

[8] If R_{zt} is measured independently of R_{mt} and equation (9) correctly specifies the measurement error on R_{zt}, then the results obtained in this section is still valid. The only result needed to change is $Var(\eta_t - \delta_t) = \sigma_\eta^2 + \sigma_\delta^2$ since $Cov(\eta_t, \delta_t) = 0$.

errors on R_{mt} and R_{zt}. However, the estimated Sharpe's performance measure is not influenced by the measurement errors on R_{mt} and R_{zt} because it does not depend on the use of proxies for R_{mt} and R_{zt}. In the following two sections, the effect of the measurement errors on the estimated Treynor's and the estimated Jensen's measures is investigated.

III. The Effect of Measurement Error on Treynor's Performance Measure

To study the effect of measurement error on the ranking of Treynor's measure, the expected value of the estimated Treynor's measure derived from the two-factor model is obtained first and shown to be biased. Chen and Lee (1979) have derived the expected value of the estimated Treynor's measure associated with the Sharpe-Lintner CAPM. Following a similar analysis, the expected value of the estimated Treynor's measure associated with the time-series equation of the two-factor model defined in (2) can be easily shown to be

$$E\left(\frac{\bar{R}_j - \bar{R}_z}{\hat{\beta}_j}\right) = \left(\frac{\mu_j - \mu_z}{\beta_j}\right) \cdot e_\beta, \quad (14)$$

[9]

where R_{jt} and R_{zt} are assumed to be normally, independently distributed,

$\hat{\beta}_j$ is the least-squares estimate of the systematic risk, β_j, and is assumed to take value in a positive range, say from .0001 to 12,[10]

[9] For the detailed derivations of (14) and (15), see Chen and Lee (1979).

[10] The estimated Treynor's measure is the ratio of two normal random varibles $(\bar{R}_j - \bar{R}_z)$ and $\hat{\beta}_j$. This ratio has no finite moments, for the $\hat{\beta}_j$ in the numerator of the estimated Treynor's measure can take value zero. Since all portfolios have positive betas and very few securities have negative betas; it is reasonable to assume that betas take positive values only. The truncated value of $\hat{\beta}_j$ in a positive range will assure the existence of the expected value of the estimated Treynor's measure. For a detailed discussion of the ratio of two normal random variables, see Hinkley (1969).

$\mu_j = E(R_{jt})$, $\mu_z = E(R_{zt})$, $\bar{R}_j = \sum_{t=1}^{n} R_{jt}/n$,

$\frac{\mu_j - \mu_z}{\beta_j}$ = ex-ante Treynor's measure,

$$e_{\beta_j} = \frac{\beta_j}{g\sqrt{2\pi}(m^*\sigma_{\tilde{\beta}} + \beta_j)} \cdot \sum_{k=0}^{\infty} [(\frac{-\sigma_{\tilde{\beta}}^2}{m^*\sigma_{\tilde{\beta}} + \beta_j})^k \sum_{\ell=1}^{k} \binom{k}{\ell} (-m^*)^{k-\ell} \cdot$$

$$2^{(\ell+1)/2}\,\Gamma(\frac{\ell + 1}{2}) m_\ell], \qquad (15)$$

$g = \int_a^b f(\tilde{\beta}_j)d\tilde{\beta}_j$, $f(\tilde{\beta}_j)$ = the probability density function of $\tilde{\beta}_j$,

$$m_\ell = \int_{a'^2}^{b'^2} \frac{X^{\frac{(\ell + 1)}{2} - 1} e^{-x/2}}{\Gamma(\frac{d + 1}{2}) 2^{(\ell+1)/2}} dx,$$

$a' = \frac{a - \beta_j}{\sigma_{\tilde{\beta}_j}}$, $b' = \frac{b - \beta_j}{\sigma_{\tilde{\beta}_j}}$, $\sigma_{\tilde{\beta}} = \sqrt{Var(\tilde{\beta}_j)}$, and $m^* = \frac{1}{2}(a' + b')$.

Equation (14) indicates that the sample representation of ex-ante Treynor's measure, $(\bar{R}_j - \bar{R}_z)/\tilde{\beta}_j$, is not an unbiased estimator.[11] And the bias factor, e_{β_j}, in (15) associated with the expected value of the $(\bar{R}_j - \bar{R}_z)/\tilde{\beta}_j$ is unique to a given security (or portfolio). Thus, the ranking of Treynor's measure based on the sample estimate, $(\bar{R}_j - \bar{R}_z)/\tilde{\beta}_j$, is not an unbiased representation of the ranking of ex-ante Treynor's measure. In other words, the ranking of Treynor's measure based on the order of the magnitudes of $(\bar{R}_j - \bar{R}_z)/\tilde{\beta}_j$ will be biased. The implication of this result is

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[11]Note that $(\frac{\bar{R}_j - \bar{R}_z}{\tilde{\beta}_j})/e_{\beta_j}$ is not an unbiased estimator of $(\mu_j - \mu_z)/\beta_j$ since e_β involves unknown parameter β_j.

that some portfolios may be evaluated favorably while others unfavorably. Furthermore, the measurement errors on R_{mt} and R_{zt} can seriously affect the ranking of Treynor's measure. This can be demonstrated analytically as follows: following a similar analysis as employed by Chen and Lee (1979), it is easy to show that the expected value of the estimated Treynor's measure associated with the measurement errors on R_{mt} and R_{zt} can be written as

$$E\left(\frac{\bar{R}_j - \bar{R}'_z}{\hat{\beta}'_j}\right) = \left(\frac{\mu_j - \mu'_z}{\beta_j + \psi_j}\right) e_{\beta'_j}$$

$$= \left(\frac{\mu_j - \mu_z}{\beta_j + \psi_j}\right) e_{\beta'_j} + \left(\frac{A}{\beta_j + \psi_j}\right) e_{\beta'_j}$$

$$= \left(\frac{\beta_j e_{\beta'}}{\beta_j + \psi_j}\right) \left[\left(\frac{\mu_j - \mu_z}{\beta_j}\right) + \frac{A}{\beta_j} \right], \qquad (16)$$

where $\mu'_z = E(\bar{R}'_z) = E(\bar{R}_z - A - \bar{\delta}) = \mu_z - A$, $\mu_z = E(\bar{R}_z)$, $\beta'_j = \beta_j + \psi_j$ by equation (11), and $e_{\beta'_j}$ is defined in (15) with replacing β_j by β'_j.[12]

If there exist no measurement errors on R_{mt} and R_{zt}, equation (16) reduces to equation (14) since $\psi_j = 0$, $A = 0$, and $\beta'_j = \beta_j$. However, the comparison of (16) with (14) indicates that the measurement errors on R_{mt} and R_{zt} increase the biasedness of the estimated Treynor's measure, $(\bar{R}_j - \bar{R}'_z)/\hat{\beta}'_j$. This will influence the ranking in a great degree. Moreover, equation

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[12] To enable $E\left(\frac{\bar{R}_j - \bar{R}'_z}{\hat{\beta}'_j}\right)$ having a finite value, the distribution of $\hat{\beta}'_j$ is also truncated in a positive interval.

(16) can be used to show that the ranking of Treynor's measure can be inconsistent if different (market) indices are used to estimate systematic risks, β_j. Different (market) indices used as proxies for the market portfolio are associated with different degrees of measurement error. This implies that for a given index used there is an estimated systematic risk for each asset (or portfolio), and hence an estimated Treynor's measure for the asset (or portfolio). Therefore, an asset (or portfolio) may be evaluated more favorably than some other assets (or portfolios) for a given index used. On the other hand, this evaluation result may be be reversed if another index is used. This can be shown in the following.

To simplify the analysis, we let the measurement errors on R_{mt} and R_{zt} be specified by equations (3) and (9), respectively, with $\tau = 0$ and $A = 0$. Assume that two portfolios (or assets) i and j are to be evaluated. Then, a market index is chosen as a proxy for the market portfolio. Suppose that based on the market index chosen the estimated Treynor's measure of portfolio j, $\hat{T}_j'$, is greater than that of portfolio i, $\hat{T}_i'$. Then, the expected difference, $E(\hat{T}_j' - \hat{T}_i')$, is positive. Using equation (16) gives the equation

$$0 < E(\hat{T}_j' - \hat{T}_i') = D(T_j - T_i) \tag{17}$$

where $\hat{T}_j' = (\bar{R}_j - \bar{R}_z)/\hat{\beta}_j'$, $\hat{T}_i' = (\bar{R}_i - \bar{R}_z)/\hat{\beta}_i'$,

$T_j = [E(R_j)-E(R_z)]/\beta_j$ = the ex-ante Treynor's measure of portfolio j,

$T_i = [E(R_i)-E(R_z)]/\beta_i$ = the ex-ante Treynor's measure of portfolio i,

$D = \left(\frac{\beta_j e_{\beta_j'}}{\beta_j + \psi_j}\right) - \left(\frac{\beta_i e_{\beta_i'}}{\beta_i + \psi_i}\right)$, and

$e_{\beta'_j}$ and $e_{\beta'_i}$ are defined in (15) with appropriate subscripts.

Now, suppose that a second index is used for the performance evaluation of portfolios i and j. Then, by analogy of equation (17), one obtains

$$E(\hat{T}^*_j - \hat{T}^*_i) = D^*(T_j - T_i) \tag{18}$$

where $\hat{T}^*_j = (\bar{R}_j - \bar{R}^*_z)/\hat{\beta}^*_j$

= the estimated Treynor's measure of portfolio j with $\hat{\beta}^*_j$ calculated against the second index.

$\hat{T}^*_i = (\bar{R}_i - \bar{R}^*_z)/\hat{\beta}^*_i$

= the estimated Treynor's measure of portfolio i with $\hat{\beta}^*_i$ computed against the second index,

$\bar{R}^*_z$ = the average return of R_{zt} associated with the second index used,

$$D^* = \left(\frac{\beta_j e_{\beta^*_j}}{\beta_j + \psi^*_j}\right) - \left(\frac{\beta_i e_{\beta^*_i}}{\beta_i + \psi^*_i}\right),$$

ψ^*_i = the bias caused by the use of the second index in estimating β_i, and

$\beta^*_j = \beta_j + \psi^*_j$.

Using equation (17), equation (18) can be rewritten as

$$E(\hat{T}^*_j - \hat{T}^*_i) = \frac{D^*}{D} \cdot E(\hat{T}'_j - \hat{T}'_i). \tag{19}$$

If there exist no measurement errors on the both indices used, equation (19) becomes $E(\hat{T}^*_j - \hat{T}^*_i) = E(\hat{T}'_j - \hat{T}'_i) > 0$; and hence $\hat{T}^*_j > \hat{T}^*_i$,[13] the ranking remains unchanged. The implication of this result is that if two indices are chosen such that their measurement errors are negligible, the effect of the measurement errors will not significantly affect the estimated systematic risk associated with each index, and hence the estimated Treynor's measures. As a result, the ranking of Treynor's performance measure would be consistent if such two indices are selected. Therefore, judicious choice of indices (with insignificant measurement errors) will produce consistent ranking of performance evaluation.

Note that if a selected index (R'_{mt}) is mean-variance efficient, the models specifying the measurement errors on R_{mt} and R_{zt} in (3) and (9) becomes respectively as

$$R_{mt} = R'_{mt} + \tau \tag{20}$$

and

$$R_{zt} = R'_{zt} + A , \tag{21}$$

for a mean-variance efficient index contains no unsystematic variation ($\eta t = 0 = \delta t$) in accordance with capital market equilibrium theory. Then, equations (5) and (11) are equal.[14]

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[13] If there exists no measurement error, then $\psi_i = \psi_j = \psi^*_i = \psi^*_j = 0$. Thus, $\hat{\beta}'_j = \hat{\beta}^*_j$, $\hat{\beta}'_i = \hat{\beta}^*_i$, $\beta'_j = \beta^*_j = \beta_j$, $e_{\beta^*_j} = e_{\beta'_j}$, $e_{\beta^*_i} = e_{\beta'_i}$, $\overline{R}_z = \overline{R}^*_z$, $\frac{D^*}{D} = \frac{e_{\beta^*_i} - e_{\beta^*_i}}{e_{\beta'_j} - e_{\beta'_i}} = 1$, and hence $\hat{T}_j = \hat{T}'_j > \hat{T}'_i = \hat{T}^*_i$.

[14] This result is clear since $\sigma^2_\eta = 0$ and $\psi_j = 0$.

$$\hat{\beta}_j = \hat{\beta}'_j = \beta_j + s_j \ . \tag{22}$$

Thus, the systematic risk computed against a mean-variance efficient index is always equal to the ex-ante systematic risk (β_j) plus the sampling variation of β_j. Following footnote 13, this result implies that ranking of the estimated Treynor's measure based on an efficient index would be consistent with that based on another efficient index.

However, it can be shown that the measurement errors on both indices chosen can lead to inconsistent ranking of Treynor's measure. If either D or D* of equation (19) becomes negative, then $E(\hat{T}^*_j - \hat{T}^*_i) < 0$. Under this circumstance, the expected ranking of the performance of portfolios i and j is reverse, for $E(\hat{T}^*_j) < E(\hat{T}^*_i)$. The implication of the reverse expected ranking is that some realization of $\hat{T}^*_j$ is very likely to be less than that of $\hat{T}^*_i$. Specifically, due to the effect of the measurement errors, portfolio j can be evaluated to be inferior to portfolio i, a reverse ranking. Therefore, inconsistent ranking of the estimated Treynor's measure may arise if different indices are used for the performance evaluation of portfolios (or assets).

IV. The Effect of Measurement Error on Jensen's Performance Measure

In this section, the estimated Jensen's performance measure is shown to be biased if there exist measurement errors on R_{mt} and R_{zt}. Then, it is shown that the estimated Jensen's measure can also produce inconsistent ranking of the performance of portfolios (or assets) if different indices are used. The inconsistent ranking can be explained by exploring the effect of the measurement errors on R_{mt} and R_{zt}. The biasedness of the estimated Jensen's measure is first studied in the following manner.

We have known that the estimated Jensen's measure, $\hat{\alpha}_j'$, is defined in (12) if there exist measurement errors on R_{mt} and R_{zt}. Taking the expected value of (12), we obtain

$$E(\hat{\alpha}_j') = \alpha_j + \psi_j(\mu_m - \mu_z), \tag{23}$$

where $\mu_m = E(R_{mt}) = \mu_m' + \tau$, $\mu_z = \mu_z' + A$, and $E(s_j) = 0$.

If there exist no measurement errors on R_{mt} and R_{zt} equation (23) becomes $E(\hat{\alpha}_j') = \alpha_j$ since $\psi_j = 0$. Then, the estimated Jensen's measure is an unbiased estimate of ex-ante Jensen's measure, α_j. Thus, the ranking of the performance of portfolios (or assets) based on $\hat{\alpha}_j'$ is an unbiased ranking of Jensen's measure. However, as indicated by (23), if there exist measurement errors on R_{mt} and R_{zt}, the estimated Jensen's measure is biased with a bias factor, $\psi_j(\mu_m - \mu_z)$.[15] The expected value of $\hat{\alpha}_j'$ is biased upward or downward, depending on ψ_j and the expected excess market return, $\mu_m - \mu_z$. Under the market condition of μ_m being greater than μ_z, the $E(\hat{\alpha}_j')$ is biased upward (downward) for a positive (negative) ψ_j which represents the effect of the measurement errors. If μ_m is less than μ_z, the $E(\hat{\alpha}_j')$ is biased downward (upward) for a positive (negative) effect, ψ_j. Therefore, if there exist measurement errors on R_{mt} and R_{zt}, the ranking based on the magnitudes of $\hat{\alpha}_j'$ will not be an unbiased representation of the true ranking of Jensen's performance measure.

The choice of different indices as a proxy for the market portfolio is critical to the consistency of the ranking of Jensen's performance

[15] For a similar reason as indicated in footnote 9, $[\hat{\alpha}_j' - \psi_j(\mu_m - \mu_z)]$ can not be an unbiased estimator of α_j.

measure. Suppose that a (market) index is used for the performance evaluation of portfolios i and j. Let the estimated Jensen's measures, associated with the chosen index, of portfolios i and j be $\hat{\alpha}'_i$ and $\hat{\alpha}'_j$, respectively, such that $\hat{\alpha}'_j > \hat{\alpha}'_i > 0$. Then, using equation (23) it can be shown that

$$0 < E(\hat{\alpha}'_j - \hat{\alpha}'_i) = (\alpha_j - \alpha_i) + (\psi_j + \psi_i)(\mu_m - \mu_z). \qquad (24)$$

Now, suppose that another index is used for the purpose of performance evaluation. Then, the expected value of the difference between the estimated Jensen's measure of portfolio j, $\hat{a}'_j$, and that of portfolio i, $\hat{a}'_i$, based on the second index can be written as

$$E(\hat{a}'_j - \hat{a}'_i) = (\alpha_j - \alpha_i) + (\psi^*_j - \psi^*_i)(\mu_m - \mu_z), \qquad (25)$$

where ψ^*_j and ψ^*_i represent the effects of the measurement errors induced by the second index in estimating Jensen's measures for portfolios j and i, respectively. Using equations (24) and (25) gives

$$E(\hat{a}'_j - \hat{a}'_i) = (\alpha_j - \alpha_i) + \frac{\psi^*_j - \psi^*_i}{\psi_j - \psi_i}[E(\hat{\alpha}'_j - \hat{\alpha}'_i) - (\alpha_j - \alpha_i)]. \qquad (26)$$

If the relative differential effect, $(\psi^*_j - \psi^*_i)/(\psi_j - \psi_i)$, of the measurement errors on both chosen indices is negative, an inequality relation can be established by using (26):

$$E(\hat{a}'_j - \hat{a}'_i) < (\alpha_j - \alpha_i)[\frac{\psi^*_j - \psi^*_i}{\psi_j - \psi_i} - 1], \qquad (27)$$

since $\frac{\psi_j^* - \psi_i^*}{\psi_j - \psi_i} \cdot E(\hat{\alpha}_j' - \hat{\alpha}_i') < 0$ in (26). If α_j is less than α_i,[16] then $E(\hat{a}_j' - \hat{a}_i') < 0$. This result implies that if the relative differential effect of the measurement errors is negative, the estimated Jensen's measure of portfolio j, $\hat{a}_j'$, is likely to be less than that of portfolio i, $\hat{a}_i'$, for $E(\hat{a}_j') < E(\hat{a}_i')$. In this case, the ranking of $\hat{a}_j'$ and $\hat{a}_i'$ is not identical to the ranking of $\hat{\alpha}_j'$ and $\hat{\alpha}_i'$. Thus, the above analysis has shown that the effects of the measurement errors associated with different indices used in performance evaluation can produce inconsistent ranking of Jensen's performance measure. This finding adds to the problem associated with Roll's (1978) finding that ambiquity arises in performance evaluation when Jensen's measure is estimated by return generating models associated with different indices used as proxies for a common generating factor.

It should be noted that if two chosen indices have no (or insignificant) measurement errors, then $E(\hat{a}_j' - \hat{a}_i') = E(\hat{\alpha}_j' - \hat{\alpha}_i')$ and $\hat{a}_j' = \hat{\alpha}_j' > \hat{\alpha}_i' = \hat{a}_i'$. Thus, the ranking of Jensen's performance measure is consistent despite different indices are used for performance evaluation. Moreover, if a chosen index is mean-variance efficient, following equations (20) and (21), it can be shown that equations (6) and (12) are identical

$$\hat{\alpha}_j = \hat{\alpha}_j' = \alpha_j - s_j(\overline{R}_m - \overline{R}_z). \qquad (28)$$

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[16]In fact, following (21), it is reasonable to assume that $\alpha_j > \alpha_i$ though the order of the magnitudes of α_j and α_i is not known.

Thus, the estimated Jensen's measure ($\hat{\alpha}_j$) is always ex-ante Jensen's measure (α_j) minus the sampling variation of β_j times the average excess market return, ($\overline{R}_m - \overline{R}_z$) if a selected index is mean-variance efficient. As a result, if two chosen indices are mean-variance efficient, equations (24) and (25) are equal:

$$0 < E(\hat{\alpha}_j' - \hat{\alpha}_i') = E(\hat{a}_j' - \hat{a}_i') = \alpha_j - \alpha_i \ , \tag{29}$$

since $\eta_t = \delta_t = 0$ and hence $\psi_i = \psi_j = \psi_j^* = \psi_i^* = 0$. Equation (29) implies that ranking of Jensen's measure is consistent for different mean-variance efficient indices used as proxies for the market portfolio.

V. Summary

The measurement errors on both market rates of return and the return on the minimum-variance zero beta portfolio have been shown to have a significant impact on the estimated alpha and the estimated systematic risk. The measurement errors will also upward (downward) bias performance evaluation of portfolios (or assets). As a result, they can lead to inconsistent ranking of the performance evaluation. However, judicious choice of indices which have insignificant measurement errors can yield an acceptable performance evaluation of portfolios (or assets).

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Appendix

(A). The derivation of equations (5) and (6):

Following equation (4) and Johnston (1972), the presumed model associated with the measurement error on R_{mt} can be written as

$$R_{jt} - R_{zt} = \alpha_j + \beta_j[(R'_{mt} + \tau + \eta_t) - R_{zt}] + [\varepsilon_{jt} - \beta_j(\tau + \eta_t)], \quad \text{(A.1)}$$

To facilitate the analysis, (A.1) is reparametrized as

$$R_{jt} - R_{zt} = \alpha^*_j + \beta_j(\gamma_{mt} + \eta'_t - \gamma_{zt}) + [\varepsilon_{jt} - \beta_j(\tau + \eta_t)], \quad \text{(A.2)}$$

where $\gamma_{mt} = R'_{mt} - \overline{R}'_m$, $\gamma_{zt} = R_{zt} - \overline{R}_z$, $\eta'_t = \eta_t - \overline{\eta}$,

and $\alpha^*_j = \alpha_j + \beta_j(\overline{R}'_m + \tau + \overline{\eta} - \overline{R}_z)$.

Then, (A.2) can be written in a matrix equation:

$$\underline{y} = \underline{\overline{x}\beta} + (\underline{\varepsilon} - v\underline{\beta}), \quad \text{(A.3)}$$

where $\underline{y}' = (R_{j1} - R_{z1}, R_{j2} - R_{z2}, \ldots, R_{jn} - R_{zn})$,

$\underline{\beta}' = (\alpha^*_j, \beta_j)$,

$$\underset{(nx2)}{\overline{\underline{x}}} = \begin{pmatrix} 1 & (\gamma_{m1} + \eta'_1 - \gamma_{z1}) \\ 1 & (\gamma_{m2} + \eta'_2 - \gamma_{z2}) \\ \vdots & \vdots \\ 1 & (\gamma_{mn} + \eta'_n - \gamma_{zn}) \end{pmatrix}, \quad \underset{(nx2)}{v} = \begin{pmatrix} 0 & \tau + \eta_1 \\ 0 & \tau + \eta_2 \\ \vdots & \vdots \\ 0 & \tau + \eta_n \end{pmatrix}$$

$\underline{\varepsilon}' = (\varepsilon_{j1}, \varepsilon_{j2}, \ldots, \varepsilon_{jn})$, and

$(\underline{\varepsilon} - v\underline{\beta})$ is a column whose typical element is $[\varepsilon_{jt} - \beta_j(\tau + \eta_t)]$, $t = 1, 2, \ldots, n$.

Thus, the least-squares estimator of $\underline{\beta}$ in (A.3) is

$$\hat{\underline{\beta}} = \underline{\beta} + (\overline{\underline{x}}'\overline{\underline{x}})^{-1}\overline{\underline{x}}'(\underline{\varepsilon} - v\underline{\beta}), \tag{A.4}$$

where

$$(\overline{\underline{x}}'\overline{\underline{x}})^{-1} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sum_{t=1}^{n} (\gamma_{mt} + \eta'_t - \gamma_{zt})^2} \end{pmatrix}$$

The estimate of β_j can be rewritten explicitly from (A.4) as follows:

$$\hat{\beta}_j = \beta_j + \frac{\sum_{t=1}^{n} [\varepsilon_{jt} - \beta_j(\tau + \eta_t)](\gamma_{mt} + \eta'_t - \gamma_{zt})}{\sum_{t=1}^{n} (\gamma_{mt} + \eta'_t - \gamma_{zt})^2} \tag{A.5}$$

Taking the probability limit of (A.5), we have

$$\operatorname*{plim}_{n\to\infty} \hat{\beta}_j = \beta_j - \frac{\beta_j\sigma_\eta^2}{\sigma_m^2 + \sigma_\eta^2 + \sigma_z^2}$$

Since $\hat{\beta}_j = \operatorname*{plim}_{n\to\infty} \hat{\beta}_j + s_j$, then $\hat{\beta}_j$ can be rewritten as

$$\hat{\beta}_j = \beta_j - \left(\frac{\beta_j\sigma_\eta^2}{\sigma_m^2 + \sigma_\eta^2 + \sigma_z^2}\right) + s_j, \tag{A.5}$$

where s_j indicates the sample variation of β_j. Equation (A.5) is equation (5). Similarly,

$$\hat{\alpha}_j = (\overline{R}_j - \overline{R}_z) - \hat{\beta}_j(\overline{R}'_m + \tau + \overline{\eta} - \overline{R}_z)$$

$$= (\overline{R}_j - \overline{R}_z) - (\operatorname*{plim}_{n\to\infty} \hat{\beta}_j + s_j)(\overline{R}'_m + \tau + \overline{\eta} - \overline{R}_z)$$

$$= \alpha_j - \left(s_j - \frac{\beta_j \sigma_\eta^2}{\sigma_m^2 + \sigma_\eta^2 + \sigma_z^2}\right)(\overline{R}'_m + \tau + \overline{\eta} - \overline{R}_z), \qquad (A.6)$$

where $\alpha_j = (\overline{R}_j - \overline{R}_z) - \beta_j(\overline{R}'_m + \tau + \overline{\eta} - \overline{R}_z)$, $\overline{R}_j = \sum_{t=1}^{n} R_{jt}/n$.

Equation (A.6) is equation (6).

(B). The derivation of Equations (11) and (12):

Following equation (10), the presumed model associated with the measurement errors on R_{mt} and R_{zt} is

$$R_{jt} - (R'_{zt} + A + \delta_t) = \alpha_j + \beta_j[(R'_{mt} + \tau + \eta_t) - (R'_{zt} + A + \delta_t)]$$
$$+ [\varepsilon_{jt} + (A + \delta_t) - \beta_j(\tau + \eta_t - A - \delta_t)] \qquad (B.1)$$

The reparametrization of (B.1) gives

$$R_{jt} - (R'_{zt} + A + \delta_t) = \alpha_j^{**} + \beta_j(\gamma_{mt} + \eta'_t - \gamma_{zt} - \delta'_t)$$
$$+ [\varepsilon_{jt} + (A + \delta_t) - \beta_j(\tau + \eta_t - A - \delta_t)], \qquad (B.2)$$

where $\alpha_j^{**} = \alpha_j + \beta_j[(\overline{R}'_m + \tau + \overline{\eta}) - (\overline{R}'_z + A + \overline{\delta})]$, $\gamma_{zt} = R'_{zt} - \overline{R}'_z$,

$\delta'_t = \delta_t - \overline{\delta}$, $\overline{R}'_z = \sum_{t=1}^{n} R'_{zt}/n$, and $\overline{\delta} = \sum_{t=1}^{n} \delta_t/n$.

Then, equation (B.2) can be written in a matrix form:

$$\underline{y}^* = \underline{\overline{x}^*}\underline{\beta}^* + (\underline{\varepsilon} + \underline{u} - v^*\underline{\beta}^*) \qquad (B.3)$$

where $\underline{y}^*$ = a column vector whose typical element is $[R_{jt} - (R'_{zt} + A + \delta_t)]$,

$\underline{\beta}^{*\prime} = (\alpha_j^{**}, \beta_j)$,

$\underline{u}$ = a column vector whose typical element is $(A + \delta_t)$,

$$\underline{\bar{x}}^* = \begin{pmatrix} 1 & (\gamma_{m1} + \eta_1') - (\gamma_{z1} + \delta_1') \\ 1 & (\gamma_{m2} + \eta_2') - (\gamma_{z2} + \delta_2') \\ \vdots & \vdots \\ 1 & (\gamma_{mn} + \eta_n') - (\gamma_{zn} + \delta_n') \end{pmatrix} \qquad v^* = \begin{pmatrix} 0 & (\tau + \eta_1 - A - \delta_1) \\ 0 & (\tau + \eta_2 - A - \delta_2) \\ \vdots & \vdots \\ 0 & (\tau + \eta_n - A - \delta_n) \end{pmatrix}$$

and $(\underline{\varepsilon} + \underline{u} - v^*\underline{\beta}^*)$ is a column vector whose typical element is $[\varepsilon_{jt} + (A + \delta_t) - \beta_j(\tau + \eta_t - A - \delta_t)]$. Then, the least-squares estimator of $\underline{\beta}^*$ in (B.3) is

$$\hat{\underline{\beta}}^* = \underline{\beta} + (\underline{\bar{x}}^{*\prime}\underline{\bar{x}}^*)^{-1}\underline{\bar{x}}^{*\prime}(\underline{\varepsilon} + \underline{u} - v^*\underline{\beta}^*), \tag{B.4}$$

where

$$(\underline{\bar{x}}^{*\prime}\underline{\bar{x}}^*)^{-1} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\sum_{t=1}^{n} (\gamma_{mt} + \eta_t' - \gamma_{zt} - \delta_t')^2} \end{pmatrix} . \tag{B.5}$$

Thus, the estimate, $\hat{\beta}_j'$, of β_j can be obtained from (B.4):

$$\hat{\beta}_j' = \beta_j + \frac{\sum_{t=1}^{n} \{(\gamma_{mt} + \eta_t' - \gamma_{zt} - \delta_t')[\varepsilon_{jt} + (A + \delta_t) - \beta_j(\tau + \eta_t - A - \delta_t)]}{\sum_{t=1}^{n} (\gamma_{mt} + \eta_t' - \gamma_{zt} - \delta_t')^2} \tag{B.6}$$

Taking the probability limit of (B.6), we have

$$\operatorname*{plim}_{n\to\infty} \hat{\beta}'_j = \beta_j + \frac{(1 + 2\beta_j)\mathrm{Cov}(\eta_t,\delta_t) - \beta_j\sigma_\eta^2 - \sigma_\delta^2(1+\beta_j)}{\sigma_m^2 + \sigma_z^2 + \sigma_1^2}$$

$$= \beta_j + \psi_j, \qquad \text{(B.7)}$$

where $\mathrm{Cov}(\eta_t,\delta_t) = d\sigma_\eta^2$, $\sigma_1^2 = \mathrm{Var}(\eta_t - \delta_t)$, and

$$\psi_j = \frac{\sigma_\eta^2[d(1 + 2\beta_j) - \beta_j - c(1 + \beta_j)]}{\sigma_m^2 + \sigma_z^2 + \sigma_1^2}$$

then, $$\hat{\beta}_j = \operatorname*{plim}_{n\to\infty} \hat{\beta}_j + s_j = \beta_j + \psi_j + s_j, \qquad \text{(B.8)}$$

which is equation (11). Following a similar analysis of obtaining equation (A.6), we have

$$\hat{\alpha}'_j = \alpha_j + (\psi_j - s_j)(\overline{R}'_m + \tau + \overline{\eta} - \overline{R}'_z - A - \overline{\delta}), \qquad \text{(B.9)}$$

which is equation (12).